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Entropy of Spin Fields in Schwarzschild Spacetime

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By using the Teukolsky master equation, we consider the gravitational, electromagnetic, and neutrino fields in Schwarzschild spacetime. The free energy and entropy of the spin fields are obtained in terms of the brick-wall model. It is shown that the entropy of all the spin fields due to the presence of the event horizon is proportional to the surface area of the event horizon, and the entropy of the neutrino field is the absolute minimum.

To find a statistical origin of the black hole entropy by employing the so-called brick-wall model, 't Hooft (1985) studied first the statistical mechanics of a free scalar field propagating in a Schwarzschild black hole background. Subsequently, quantum corrections to the Bekenstein–Hawking entropy due to a scalar field have been studied by different authors in different types of spacetime, such as the Reissner–Nordström (Demers *et al.*, 1995), Reissner–Nordström–(anti-)de Sitter (Cai and Zhang, 1996), and BTZ (Kim, 1999) spacetimes. In the present paper we focus our attention on the gravitational, electromagnetic, and neutrino fields in a Schwarzschild background. We briefly review the Teukolsky master equation and then compute the entropy of spin fields by using the brick-wall model.

In the case of a Schwarzschild black hole, spacetime is described by the metric

$$g_{\mu\nu} = \operatorname{diag}\left[\frac{\Delta}{r^2}, -\frac{r^2}{\Delta}, -r^2, -r^2\sin^2\theta\right]_{\mu\nu} \tag{1}$$

where $\Delta = r(r - 2M)$, and M is the mass of the black hole.

Teukolsky (1973), using the Newman–Penrose formalism, succeeded in disentangling the perturbations of the Kerr metric, and derived a master

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equation governing not only gravitational perturbations, but electromagnetic, neutrino, and scalar fields as well. In the Schwarzschild limit, this master equation reads

$$\left[\frac{r^4}{\Delta}\frac{\partial^2}{\partial t^2} + \frac{2r^2 p(r-3M)}{\Delta}\frac{\partial}{\partial t} - \Delta\frac{\partial^2}{\partial r^2} - 2(p+1)(r-M)\frac{\partial}{\partial r} - \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) - \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2} - \frac{2ip\cos\theta}{\sin^2\theta}\frac{\partial}{\partial\varphi} + p^2\cot^2\theta - p)\right]\Psi_p = 0 \tag{2}$$

where p is the helicity of the field. $p = \pm 2$, $\Psi_2 \equiv \Psi_0^B$, and $\Psi_{-2} \equiv r^4\Psi_4^B$ for the gravitational field; $p = \pm 1$, $\Psi_1 \equiv \Phi_0$, and $\Psi_{-1} \equiv r^2\Phi_2$ for the electromagnetic field; $p = -1/2$ and $\Psi_{-1/2} = r\chi_1$ for the neutrino field; and $p = 0$ and $\Psi_0 = \Phi$ for the scalar field.

By separating variables in Eq. (2), we obtain the following complete set of solutions:

$$\Psi_p = e^{-iEt}\,{}_pR_{lE}(r)\,{}_pY_l^m(\theta,\varphi) \tag{3}$$

where l and m are integers satisfying the inequalities $l \geq |p|$ and $-l \leq m \leq l$, and ${}_pY_l^m(\theta,\varphi)$ is a spin-weighted spherical harmonic (Goldberg *et al.*, 1967). ${}_pR_{lE}(r)$ satisfies the ordinary differential equation

$$\left[\Delta^{-p}\frac{d}{dr}\left(\Delta^{p+1}\frac{d}{dr}\right) + \frac{r^4E^2 + 2ipEr^2(r-3M)}{\Delta} - (l-p)(l+p+1)\right]{}_pR_{lE}(r) = 0 \tag{4}$$

In the WKB approximation one writes ${}_pR_{lE}(r) = {}_p\Gamma_{lE}(r)\exp[iS(r, p, l, E)]$, where ${}_p\Gamma_{lE}(r)$ is a slowly varying amplitude and $S(r, p, l, E)$ is a rapidly varying phase. To leading order, only first derivatives of the phase are important. In particular, Eq. (4) yields the radial wave number $k(r, p, l, E) \equiv \partial_r S$:

$$k^2 = \frac{1}{\Delta}\left[\frac{r^4}{\Delta}E^2 - (l-p)(l+p+1)\right] \tag{5}$$

There is an ultraviolet cutoff just outside the event horizon r_H $(=2M)$ and an infrared cutoff for a large distant L in the brick-wall model of 't Hooft (1985). According to the semiclassical quantization rule, the radial wave number is quantized as

$$n\pi = \int_{r_H+\varepsilon}^{L} dr\, k(r, p, l, E) \tag{6}$$

where ε is a small, positive constant. Then, the number of eigenstates with energy smaller than E is given by

$$\begin{aligned} g(E) &= \sum_p \sum_l (2l + 1)n \\ &= \sum_p \sum_l (2l + 1)\frac{1}{\pi}\int_{r_H+\varepsilon}^{L} dr\, k(r, p, l, E) \\ &= \frac{1}{\pi}\sum_p \sum_l (2l + 1)\int_{r_H+\varepsilon}^{L}\left[\frac{r^4}{\Delta^2}E - \frac{1}{\Delta}(l - p)(l + p + 1)\right]^{1/2} dr \\ &= \frac{1}{\pi}\sum_p \int_{r_H+\varepsilon}^{L} dr \int_{(|p|+1/2)^2}^{r^4E^2/\Delta+(p+1/2)^2} d\eta\left\{\frac{r^4}{\Delta^2}E^2 - \frac{1}{\Delta}\left[\eta - \left(p + \frac{1}{2}\right)^2\right]\right\}^{1/2} \end{aligned} \tag{7}$$

where we have set $\eta = (l + 1/2)^2$ and $\Sigma_l(2l + 1) \rightarrow \int d\eta$. The upper limit of integration in the variable η is due to the fact that k^2 has to be positive. The integration in η can be performed and so

$$\begin{aligned} g(E) &= \frac{2}{3\pi}\sum_p \int_{r_H+\varepsilon}^{L}\frac{r^6}{\Delta^2}\left[E^2 - \frac{\Delta}{r^4}(|p| - p)\right]^{3/2} dr \\ &\approx \sum_p\left(\frac{2E^3r_H^4}{3\pi\varepsilon} + \frac{E^3V}{6\pi^2}\right) = \omega\left(\frac{2E^3r_H^4}{3\pi\varepsilon} + \frac{E^3V}{6\pi^2}\right) \end{aligned} \tag{8}$$

where in the latter expression only the leading divergences have been written down. Here $V \approx 4\pi L^3/3$, $\omega = 2$ corresponds to the gravitational and electromagnetic fields, and $\omega = 1$ corresponds to the neutrino and scalar fields.

In terms of the standard statistical mechanics, the free energy of spin fields at the inverse temperature β is given by

$$-\beta F = \pm\sum_a \ln(1 \pm e^{-\beta E_a}) \tag{9}$$

where the plus sign in Eq. (9) corresponds to the Fermi case, while the minus sign corresponds to the Bose case. Using Eq. (8) to determine the density of states, we obtain

$$F \approx \mp\frac{1}{\beta}\int_0^\infty dE\,\frac{dg(E)}{dE}\ln(1 \pm e^{-\beta E}) = -\int_0^\infty \frac{g(E)}{e^{\beta E} \pm 1}\,dE$$

$$= \begin{cases} -\dfrac{\omega\pi^2 V}{90\beta^4} - \dfrac{2\omega\pi^3 r_H^4}{45\varepsilon\beta^4} & \text{(Bose case)} \\ -\dfrac{7\omega\pi^2 V}{720\beta^4} - \dfrac{7\omega\pi^3 r_H^4}{180\varepsilon\beta^4} & \text{(Fermi case)} \end{cases} \tag{10}$$

The first term on the right-hand side of the latter equation is the usual contribution from the vacuum surrounding the system at large distances, while the second term is an intrinsic contribution from the event horizon and diverges linearly as $\varepsilon \to 0$.

From Eq. (10) one can easily obtain the corresponding entropy of the spin fields

$$S = \beta^2\frac{\partial F}{\partial\beta} = \begin{cases} \dfrac{2\omega\pi^2 V}{45\beta^3} + \dfrac{8\omega\pi^3 r_H^4}{45\epsilon\beta^3} & \text{(Bose case)} \\ \dfrac{7\omega\pi^2 V}{180\beta^3} + \dfrac{7\omega\pi^3 r_H^4}{45\varepsilon\beta^3} & \text{(Fermi case)} \end{cases} \tag{11}$$

At the equilibrium temperature $T = (4\pi r_H)^{-1}$, the entropy reads

$$S = \begin{cases} \dfrac{2\omega\pi^2 V}{45\beta^3} + \dfrac{\omega A}{360\pi l_p^2} & \text{(Bose case)} \\ \dfrac{7\omega\pi^2 V}{180\beta^3} + \dfrac{7\omega A}{2880\pi l_p^2} & \text{(Fermi case)} \end{cases} \tag{12}$$

where $A = 4\pi r_H^2$ is the surface area of the event horizon, and $l_p = \int_{r_H}^{r_H+\varepsilon}\sqrt{-g_{rr}}\,dr \approx 2\sqrt{\varepsilon r_H}$ is the proper distance from the event horizon r_H to $r_H + \varepsilon$.

In conclusion, by using the brick-wall model, we have obtained the free energy and entropy of the spin fields on the background of the Schwarzschild black hole; they are given by Eqs. (10) and (12). The leading terms of the free energy and entropy due to the gravitational and electromagnetic fields are equal to two times the one due to the scalar field, where the factor of two is due to the helicity of fields. The leading terms of the free energy and entropy are equal to 7/8 times the scalar one for the neutrino field. From Eq. (12) we find that the entropy of all the spin fields due to the presence of the event horizon is proportional to the surface area of the event horizon, and that the entropy of the neutrino field is the absolute minimum.

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